Theorem (Harnack inequality) Let u be a positive sol^n to the heat equation. Then

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-\frac{n}{2}} \exp\left(-\frac{|x_2-x_1|^2}{4(t_2-t_1)}\right).$$

We first state and prove the following Lemma.

Lemma:- For u as above

$$\Delta \log u + \frac{n}{2t} \geq 0.$$

Proof. First of all note that if $u=\rho=$ fundam-ental sol^n, then $\Delta \log \rho + \frac{n}{2t} = 0.$

For proving the lemma, we use the weak max. principle for supersolutions.

Let $Q = \Delta \log u$ and

Set $P = 2tQ+n$, note $P(0) = n > 0$.

We first calculate

$$(\partial_t - \Delta)\Delta \log u = \Delta(\partial_t - \Delta)\log u$$

(note commuting derivatives)

$$= \Delta\left(\frac{\partial_t u}{u} - \nabla_i\left(\frac{1}{u}\nabla_i u\right)\right)$$

$$= \Delta\left(\frac{\partial_t u}{u} - \frac{u\Delta u - |\nabla u|^2}{u^2}\right)$$

$$= \Delta\left(\frac{|\nabla u|^2}{u^2}\right) = \Delta|\nabla \log u|^2$$

$$= \Delta\langle \nabla \log u, \nabla \log u\rangle = 2\langle \nabla \log u, \nabla Q\rangle + 2|\nabla^2 \log u|^2$$

$$\geq 2\langle \nabla \log u, \nabla Q\rangle + \frac{2}{n}Q^2$$

(Cauchy-Schwarz)

$$\partial_t P = 2Q + 2t\partial_t Q, \quad \Delta P = 2t\Delta Q.$$

$$\Rightarrow (\partial_t - \Delta)P = 2Q + 2t\partial_t Q - 2t\Delta Q$$

$$\geq 2t\left(2\langle \nabla \log u, \nabla Q\rangle + \frac{2}{n}Q^2\right) + 2Q$$

$$= 2\langle \nabla \log u, \nabla P\rangle + \frac{2Q}{n}P.$$

$$\Rightarrow P \geq 0 \Rightarrow 2tQ + n \geq 0 \Rightarrow \Delta \log u + \frac{n}{2t} \geq 0.$$

$\square$

Proof of the theorem

From the calculation above, we get

$$(\partial_t - \Delta)\log u = \frac{|\nabla u|^2}{u^2}.$$

For $(x_1, t_1), (x_2, t_2)$, let $\gamma : [t_1, t_2] \to \mathbb{R}^n$ be a path from x_1 to x_2. Then

$$\frac{d}{dt}\log(\gamma(t), t) = \partial_t \log u + \frac{\nabla_{\gamma'} u}{u}$$

$$= \Delta \log u + \frac{|\nabla u|^2}{u^2} + \frac{\nabla_{\gamma'} u}{u}$$

$$\geq -\frac{n}{2t} + \frac{|\nabla u|^2}{u^2} + \frac{|\nabla u|}{|u|}|\gamma'|$$

(from the Lemma above)

$$\geq -\frac{n}{2t} - \frac{|\gamma'|^2}{4} \quad \text{(Young's ineq.)}$$

which on integration b/w t_1 and t_2 gives

$$\log \frac{u(x_2, t_2)}{u(x_1, t_1)} \geq -\frac{n}{2}\log\left(\frac{t_2}{t_1}\right) - \frac{1}{4}\int_{t_1}^{t_2} |\gamma'(t)|^2 \, dt$$

Choosing γ to be the straight line from x_1 to x_2 gives the result. $\square$